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VISCOELASTIC PROPERTIES OF FRACTAL MEDIA

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Temporal fractal sets for analysis of viscoelastic properties of nonhomogeneous media are considered. A fractional derivative directly related to fractal dimension is constructed. The relationship between the diffusion of the relaxation spectrum and the fractal dimension is established.

Introduction. The objective of the present work is to determine the relationship between the fractal dimension of a set, the fractional derivative, and the diffusion of the relaxation spectrum of a nonhomogeneous structure. Determining this relationship makes it possible to extend the physical interpretation of known experimental data obtained from studies of the relaxation properties of nonhomogeneous media.

Various systems that are superensembles consisting of hierarchically subordinate statistical ensembles have been consistently described within the framework of fractal models [1–7].

The concept of a fractal set, i.e., a set with fractional dimension, which was introduced by Mandelbrot [1] at the beginning of 1960s, has been widely used in various areas of the physics of condensed matter [2–7]. Important results have been obtained in describing systems with large fluctuations and stochastic structure [5–7].

The main procedure for obtaining a fractal set is as follows: from a set M of whole dimension d, its parts are removed under a particular law. The parts of the set are removed (replaced) by an iterative process. The set obtained is such a manner is self-similar with fractional dimension d_f. For example, a Cantor set is obtained as follows: a segment of unit length is divided into three equal parts, the middle part is removed, and each of the remaining two segments is divided again into three equal segments, etc. (Fig. 1). Performing this procedure n times ($n \to \infty$), we obtain a set which is called a "Cantor set" or Cantor "dust."

An important characteristic of a fractal set is its dimension d_f, which can be determined from the dependence of the mass (measure) of the fractal set M_f on L^{d_f}, where L is a linear scale [1, 2].

It should be noted that a main feature of fractal structures is the dependence of their properties on L^α [1–7], where α is a constant number. In real media, this dependence is usually valid in a certain limited region (region of intermediate asymptotic behavior), which is defined as $a \leqslant L \leqslant \xi$, where a is a lattice constant (microscopic constant) and ξ is the correlation length. In the range of scales $L \gg \xi$, the medium is structurally homogeneous and can be characterized by effective properties.

Along with geometric, i.e., spatial fractals, increased attention has recently been given to temporal fractals — fractal sets of times of events, in which the next event takes place at interval τ after the previous one. Temporal fractals have been used to study the dynamics of reactions in disordered media taking into account the presence of spatial and temporal disorder [8–11].

Polymers and composites are nonhomogeneous materials with "long memory," in which the strain (stress) in a given particle at a specified time depends not only on the current values of strain, temperature,

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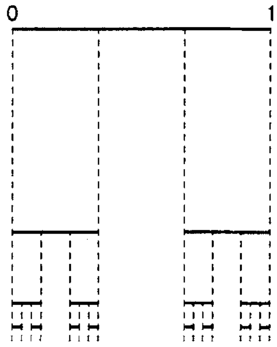

Fig. 1

and other determining parameters but also on the values of these parameters at each preceding time.

According to the Boltzmann assumption, the strain (stress) caused by the applied stress (strain), is delayed owing to the intrinsic "memory" property of the material [12].

If we assume that the strain of a medium depends on time, then, to determine the strain–stress relation, according to the Boltzmann principle, we can divide the entire time interval $(0, t)$ into n parts ($\Delta\tau_k = t_{k+1} - t_k$). Next, it is assumed that the strain of the body $\varepsilon(\tau_k)$ is a constant value in each time interval (t_k, t_{k+1}) and each strain component $\varepsilon(\tau_k)$ influences values of the stress tensor $\sigma(t)$. Thus, in each interval $\Delta\tau_k$, the stress is equal to $\sigma^k(t) = R(t, \tau_k)\varepsilon(\tau_k)\Delta\tau_k$. Here $R(t, \tau)$ is the influence coefficient (relaxation kernel), which usually has the form [13] $R(t - \tau) = C/(t - \tau)^\alpha$, where C is a constant (does not depend on time).

The Boltzmann principle implies that the total stress $\sigma(t)$ at time t is the sum of the contributions $\sigma^k(t)$ from the strain tensor in individual intervals (t_k, t_{k+1}), i.e.,

$$\sigma(t) = \sum_{k=1}^{n} \sigma^k(t) = \sum_{k=1}^{n} R(t, \tau_k)\varepsilon(\tau_k)\Delta\tau_k.$$

In this case, as $\Delta\tau_k \to 0$, the stress $\sigma(t)$ at time t is equal to

$$\sigma(t) = \int_0^t R(t, \tau)\varepsilon(\tau)\, d\tau.$$

Similarly, we can write

$$\varepsilon(t) = \int_0^t \Pi(t, \tau)\sigma(\tau)\, d\tau,$$

where $\Pi(t, \tau)$ are the influence coefficients (creep kernel). If the influence coefficients have the form

$$R(t, \tau) = C_0\delta(t - \tau), \qquad \Pi(t, \tau) = S_0\delta(t - \tau), \tag{1}$$

i.e., the medium has no "memory," Eqs. (1) becomes $\sigma = C_0\varepsilon$ and $\varepsilon = S_0\sigma$. In other words, we have Hooke's linear law, where σ and ε are the stress and strain tensors and C and S are the tensors of elastic and compliance moduli, respectively.

We consider media in which the "memory" is total but not ideal, i.e., in the interval $(0, t)$, it is preserved only in certain time intervals. When the "memory" is switched on, the δ-function in (1) is modified into a bell-shaped curve, whose width is determined by the time interval τ within which the strain (stress) depends on the stress (strain).

Two limiting processes can be distinguished in the evolution of such systems. In the first process, the system passes through all states continuously, without any losses. In the second process, some segments of the continuous states of the system are eliminated from under a specified law. This process can be characterized as a process generated by a fractal state with specified fractal dimension d_f.

To describe processes with "memory," Nigmatulin [14] used a fractional integral. Analysis was performed for a Cantor set. Nigmatulin [14] assumed that in a system with specified spatial geometry, only part of states "survive" during evolution, and the others are irreversibly lost, i.e., become unattainable for the system. The loss of states was governed by the iteration procedure for obtaining a Cantor set. In [14], it is shown that in this case (in the limit $N \to \infty$), the process can be described using the fractional integral

$$\frac{d^{-q} f(x)}{dx^{-q}} = \frac{1}{\Gamma(q)} \int_0^1 (1-y)^{q-1} f(yx)\, dy,$$

where the exponent q indicates the fraction of preserved states and coincides with the fractal dimension of the set.

In [15–17], it is shown that in a description of materials with "memory," integral operators can be replaced by differential operators.

The properties of a random medium can be described using a fractional derivative [13–23]. For example, the mechanical model of a deformable body can be represented as a certain system of elastic and viscous elements [13].

For an elastic element, Hooke's law is valid:

$$\sigma = E\varepsilon \tag{2}$$

(E is Young's modulus).

For a viscous element, the rate of change in linear dimensions is given by $d\varepsilon/dt = (1/\eta)\sigma$ or

$$\sigma = \eta\, d\varepsilon/dt, \tag{3}$$

where η is the viscosity of the material.

The relation between σ and ε, which contains the elasticity law (2) and the viscosity law (3) as the limiting cases, can be written in the form (see [13, p. 125]) $\sigma(t) = K\, d^q\varepsilon(t)/dt^q$, which leads to Hooke's law for $q = 0$ and $K = E$ and Newton's viscosity law for $q = 1$ and $K = \eta$.

Thus, to describe the viscoelastic properties of media with intermediate states, it is possible to assume a fractional value for q in the strain–stress relation.

There are several definitions of fractional derivatives [18, 19]. Among those used most widely is the definition of a fractional derivative in the Riemann–Liouville sense:

$$\frac{d^q f(x)}{dx^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x \frac{f(y)\, dy}{(x-y)^{q-n+1}} \quad \text{for} \quad n-1 \leqslant q \leqslant n,$$

where $\Gamma(n)$ is the gamma function.

We consider the definition of a fractional derivative. The functions for which the full increment

$$\Delta_h f(x) = f(x+\Delta x) - f(x)$$

is representable as

$$\Delta_h f = A(\Delta x)^h + \alpha(x)(\Delta x)^h \qquad [p \lim \alpha(x) \to 0 \quad \text{for} \quad (\Delta x)^h \to 0], \tag{4}$$

can be divided into two classes:

(a) If $h = 1$ or 0, then $f(x)$ belongs in the classical set of differentiable functions;

(b) if $h \neq 1$ or $h \neq 0$ (Hölder parameter), then $f(x)$ belongs in the set of functions for which derivatives in the usual sense do not exist but a fractional derivative exists [19]:

$$\frac{d^h f(x)}{dx^h} = \lim \frac{\Delta_h f}{\Delta x^h}, \qquad \Delta x^h \to 0. \tag{5}$$

From (4) it follows that if we introduce logarithmic metric, the increment of the function $\log \Delta_h f$ is linear in the increment of the independent variable $\log \Delta x$, and, hence, we can apply the standard differential calculus.

1. Fractional Derivative and Fractal Dimension of Set. Sometimes, fractals are defined as continuous functions which have no derivatives (tangents) at any point [24]. In this connection, in an analysis of the local properties of a fractal set, it is reasonable to use a fractional derivative.

To explain the relationship between the fractional derivative and the fractal dimension, we consider the definition of the local density of a homogeneous set. The increment in weight for a set Ω of dimension d is defined by $\Delta M = (\Delta x)^d \rho$, where ρ is the density of the set and x are the linear dimensions of the set. If the objects are homogeneous and d is an integer, the local density is equal to

$$\rho(x) = \frac{dM(x)}{d\mu} = \lim_{\Delta\mu \to 0} \frac{\Delta M(x)}{\Delta \mu} = \lim_{\Delta x \to 0} \frac{\rho(\Delta x)^d}{(\Delta x)^d} = \rho,$$

where ρ is the density of the homogeneous object, $\Delta M = \rho(x)\Delta x^d$ is the increment in weight in the neighborhood of the point x, and $\Delta\mu = (\Delta x)^d$ is the increment in measure of the set Ω, on which the density ρ is defined.

For fractal structures obtained by removal of a certain subset from the main set Ω, the increment in weight for a fractal set M_f is equal to $\Delta M_f = \rho_f \Delta x^{d_f}$, and the local density is

$$\rho_f(x) = \frac{dM_f(x)}{d\mu} = \lim_{\Delta\mu \to 0} \frac{\Delta M_f(x)}{\Delta \mu} = \lim_{\Delta x \to 0} \frac{\rho_f(\Delta x)^{d_f}}{(\Delta x)^d} = \rho_f \Delta x^{d_f - d}. \tag{6}$$

From (6) it follows that

$$\rho_f(x) = \frac{dM_f(x)}{d\mu} = \begin{cases} 0 & \text{for} \quad d < d_f, \\ \rho_f & \text{for} \quad d = d_f, \\ \infty & \text{for} \quad d > d_f. \end{cases} \tag{7}$$

Thus, according to (7), the derivative $dM_f(x)/d\mu$ has a finite value if the increment in measure for the set $\Delta\mu$ is measured not in units $(\Delta x)^d$ but in units $(\Delta x)^{d_f}$. In addition, from (6) it follows that d_f has the meaning of the dimension of the Hausdorff–Besicovich measure [1, 2].

The result obtained can be generalized as follows. We assume that on a fractal set Ω_f, a function $f(x)$ is specified and the point $x = x_0$ and its neighborhood belong to the set Ω_f with dimension d_f. We divide the segment $[x, x_0]$ into N parts. As the unit of measurement at the nth stage, we use Δx^α:

$$(\Delta x_k^{(n)})^\alpha = (1/N_n)(x_0 - x),$$

where $N_n = j^n$, i.e., $N_n = j^n$ gives the number of segments at the nth scale level, and j is the number of blocks (ramification) that make up an elementary fractal figure (for a Cantor set, $j = 2$). Then, at the nth scale level, the length of the kth partition segment is equal to

$$\Delta x_k^{(n)} = \xi^n (x_0 - x), \tag{8}$$

where ξ is the scaling factor (the similarity factor for the set Ω_f), $\xi < 1$. The number of partition points on the segment $[x, x_0]$ at the nth stage is $m_n = 1, 2, \ldots, j^{n+1}$. This partitioning of the segment $[x, x_0]$ allows each point (element) of the fractal set to be correlated to a point of ultrametric space, whose geometric image is represented by a Keily tree [24–26].

According to the definition of the fractal dimension $d_f = \alpha$, the relation $(1/\xi)^{n\alpha} = N_n$ holds.

From (8) it follows that $\lim_{n \to \infty} \Delta x_k^{(n)} = 0$. Thus, $\Delta x_k^{(n)}$ is an infinitesimal value, i.e., as $n \to \infty$, the ultrametric space becomes continual.

In what follows, the increment in argument at the nth scale level $\Delta x_k^{(n)}$ is denoted by Δx, i.e., $\Delta x = \Delta x_k^{(n)}$. The coordinates of the partition points at the nth scale level are given by $x_k = x_0 - k\Delta x_k^{(n)} = x_0 - k\Delta x$, where $k = 0, 1, 2, \ldots, j^{n+1}$. Thus, the following equalities hold:

$$\Delta x = (x_0 - x)/(1/\xi)^n, \qquad \Delta x^\alpha = (x_0 - x)^\alpha / N_n, \qquad (1/\xi)^{n\alpha} = j^n = N_n.$$

We consider the function increment

$$\Delta_\alpha f(x) = f(x_0) - f(x_0 - \Delta x), \tag{9}$$

which will be called the first difference.

From (9) the second difference $\Delta_\alpha^2 f(x)$ can be defined as the squared operator Δ_α:

$$\Delta_\alpha^2 f(x) = \Delta_\alpha(\Delta_\alpha f(x)) = \Delta_\alpha f(x_0) - \Delta_\alpha f(x_0 - \Delta x) = f(x_0) - 2f(x_0 - \Delta x) - 2f(x_0 - 2\Delta x).$$

The third difference is obtained similarly:

$$\Delta_\alpha^3 f(x) = f(x_0) - 3f(x_0 - \Delta x) + 3f(x_0 - 2\Delta x) - 3f(x_0 + \Delta x).$$

Hence it follows that the kth difference $\Delta_\alpha^k f(x)$ is expressed in terms of the alternating binomial coefficients:

$$\Delta_\alpha^k f(x_0) = \sum_{k=0}^{m} (-1)^k C_n^k (f(x_0 - k\Delta x)), \qquad C_m^k = \frac{m!}{k!(m-k)!}, \qquad m = j^{n+1}.$$

At the same time, from the definition of Δ_α it follows that $f(x_0 - \Delta x) = f(x_0) - \Delta_\alpha f(x_0) = (1 - \Delta_\alpha) f(x_0)$, where 1 is an identity operator. Then, we can write $f(x - 2\Delta x) = (1 - \Delta_\alpha) f(x_0 - \Delta x) = (1 - \Delta_\alpha)^2 f(x_0)$. Generally, $f(x_0 - k\Delta x) = (1 - \Delta_\alpha)^k f(x_0)$. Hence it follows that $f(x) = (1 - \Delta_\alpha)^m f(x_0)$, since, according to (8), $x = x_0 - m\Delta x$, where $m = j^{n+1}$.

Expanding the binomial $(1 - \Delta_\alpha)^k$ by Newton's formula, we obtain

$$f(x) = \sum_{k=0}^{m} (-1)^k C_m^k \Delta_\alpha^k f(x_0). \tag{10}$$

The general term of the sum on the right side of (10) is brought to the form

$$C_m^k \Delta_\alpha^k f(x_0) = C_m^k \frac{\Delta_\alpha^k f(x_0)}{(\Delta x^\alpha)^k} (\Delta x^\alpha)^k$$

$$= \frac{m(m-1)\cdots(m-k+1)}{k!} \frac{\Delta_\alpha^k f(x_0)}{(\Delta x^\alpha)^k} \frac{(x_0 - x)^{\alpha k}}{N_n} = P_{mk} \frac{\Delta_\alpha^k f(x_0)}{k!(\Delta x^\alpha)^k} (x_0 - x)^{\alpha k},$$

where $P_{mk} = m(m-1)\cdots(m-k+1)/N_n$, $k = 1, 2, \ldots, m$.

Thus, (10) can be written as

$$f(x) = \sum_{k=0}^{m} (-1)^k \frac{P_{mk} \Delta_\alpha^k f(x_0)}{(\Delta x^\alpha)^k} (x_0 - x)^{\alpha k}.$$

For finite k and an infinitely large m ($m \to \infty$), for the function $f(x)$ we obtain an analog of the Taylor series:

$$f(x) = \sum_{k=0}^{\infty} \frac{j^k f^{(\alpha k)}(x_0)}{k!} (x_0 - x)^{\alpha k},$$

or

$$f(x) = \sum_{k=0}^{\infty} a_k (x_0 - x)^{\alpha k}, \tag{11}$$

where $a_k = (j^k/k!) f^{(\alpha k)}(x_0)$; $f^{(\alpha k)}(x_0) = \lim_{\Delta x \to \infty} (\Delta_\alpha^k f(x_0)/(\Delta x^\alpha)^k)$ is the kth-order fractional derivative of the function $f(x)$ at the point $x = x_0$ on the fractal set Ω_f.

The coefficients of series (11) depend not only on the kth-order fractional derivative of the function $f(x)$ at the point $x = x_0$ but also on the ramification of the fractal set j on which the function $f(x)$ is specified. From (11) it follows that the first derivative ($k = 1$) is defined by

$$\frac{d^\alpha f(x_0)}{dx^\alpha} = f^{(\alpha)}(x_0) = \lim_{\Delta x^\alpha \to 0} \frac{\Delta^\alpha f(x_0)}{\Delta x^\alpha} = \lim_{\Delta x^\alpha \to 0} \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x^\alpha},$$

which coincides with the definition (5).

The fractional derivative can be expressed in terms of an integral of the function $f(x)$ on the fractal set Ω_f. The integral is equal to the limit of the integral sum

$$\int_a^b f(x)(dx)^\alpha = \lim_{\Delta x^\alpha \to 0} \sum_{k=1}^{\infty} f(x_0 - (k-1)\Delta x)(\Delta x)^\alpha. \tag{12}$$

Examination of the function $\Phi(x) = \int_0^x (x-t)^{\alpha-1} f(t)\,dt$ shows that $\frac{d^\alpha}{(dx)^\alpha}\Phi(x) = f(x)$, i.e., $\Phi(x)$ is an analog the primitive for the function $f(x)$ and the following equation holds:

$$\int_0^x f(t)(dt)^\alpha = \int_0^x (x-t)^{\alpha-1} f(t)\,dt.$$

Thus, we determined fractional integrodifferential representations that, by construction, are related to the procedure of designing a fractal set that defines the properties of the function $f(x)$.

2. Viscoelastic Properties. To establish the relation between stresses σ and strains ε for media in which the states depend on time t and part of the states are eliminated (removed) under a particular law, we use the Boltzmann superposition principle and the fractional integrodifferential representation.

We assume that the deformation of a particle (local region in a homogeneous state of stress) begins at time $t = 0$. We divide the segment $[0, t]$ by points as described above: $\tau_0 = 0$, $\tau_1 > \tau_0$, $\tau_2 > \tau_1, \ldots, \tau_n = t > \tau_{n-1}$; the points coincide with the ends of states that were not eliminated. Thus, $\Delta\tau_n^\alpha = \tau_{n+1} - \tau_n = t/N_n$, where $N_n = j^n$ is the number of segments, $\alpha = d_f = \ln j / \ln \zeta^{-1}$ is the fractal dimension of the set, and ζ is the scaling factor (similarity parameter), which describes the decrease in size of the block (region) at each scale level [2].

Considering the strain-tensor components to be constant and equal to τ_{k-1}, τ_k on each segment $[\varepsilon_{kl}(\tau_k)]$, it is possible to assume that each value of the component $\varepsilon_{kl}(\tau_k)$ influences values of the stress-tensor component $\sigma_{ij}(\tau_k)$ at time t under the linear law $\sigma_{ij}^{(k)}(\tau_k) = C_{ijkl}\varepsilon_{kl}(\tau_k)$, where $C_{ijkl} = C_{ijkl}(t, \tau_k)$ is the influence tensor. According to the Boltzmann superposition principle

$$\sigma_{ij}(t) = \sum_{n=1}^{N_n} C_{ijkl}(t, \tau_n)\varepsilon_{kl}(\tau_n)(\Delta\tau_n)^\alpha,$$

passing to the limit $\Delta\tau_n^\alpha \to 0$, we obtain

$$\sigma_{ij}(t) = \int_0^t C_{ijkl}(t, \tau)\varepsilon_{kl}(\tau)(d\tau)^\alpha.$$

The value $\alpha = 1$ corresponds to a continuous process. The expressions for the strains ε_{kl} in terms of the stresses σ_{ij} are obtained similarly:

$$\varepsilon_{ij}(t) = \int_0^t S_{ijkl}(t, \tau)\sigma_{kl}(\tau)(d\tau)^\alpha.$$

Experiments show that under rapid loading, materials with "long memory" react instantaneously to current stresses, after which creep and relaxation begin. This indicates that the influence coefficients C_{ijkl} and S_{ijkl} (kernels) have an additive singular component (proportional to the δ-function) [21, 22], i.e,

$$C_{ijkl}(t, \tau) = C_{ijkl}^0\delta(t - \tau) + R_{ijkl}(t, \tau), \qquad S_{ijkl}(t, \tau) = S_{ijkl}^0\delta(t - \tau) + \Pi_{ijkl}(t, \tau),$$

where C^0_{ijkl} and S^0_{ijkl} are the moduli of instantaneous elasticity and compliance and $R_{ijkl}(t,\tau)$, $\Pi_{ijkl}(t,\tau)$ are the relaxation and creep kernels, respectively.

Thus,

$$\sigma_{ij}(t)=C^0_{ijkl}\varepsilon_{kl}(t)+\int_0^t R_{ijkl}(t,\tau)\varepsilon_{kl}(\tau)\,(d\tau)^\alpha,\quad \varepsilon_{ij}(t)=S^0_{ijkl}\sigma_{kl}(t)+\int_0^t \Pi_{ijkl}(t,\tau)\sigma_{kl}(\tau)\,(d\tau)^\alpha. \tag{13}$$

The differential equations with fractional derivatives that correspond to (13) have the form

$$\frac{d^\alpha\sigma}{d\tau^\alpha}=C\,\frac{d^\alpha\varepsilon}{d\tau^\alpha}+R\varepsilon,\qquad \frac{d^\alpha\varepsilon}{d\tau^\alpha}=S\,\frac{d^\alpha\sigma}{d\tau^\alpha}+n\sigma. \tag{13$'$}$$

For example, for continuous processes ($\alpha = 1$) for Maxwell and Voigt media, Eqs. (13′) become [22]

$$\frac{d\sigma}{dt}=\eta\,\frac{d\varepsilon}{dt}+\mu\varepsilon,\qquad \frac{d\varepsilon}{dt}=\frac{1}{\mu}\frac{d\sigma}{dt}+\frac{1}{\eta}\,\sigma,$$

where η is the viscosity of the medium and μ is the shear modulus.

For a Ziner medium [22], which combines Maxwell and Voigt media, the equation becomes

$$\sigma+\tau_\varepsilon\,\frac{d^\alpha\sigma}{dt^\alpha}=\mu\Big(\varepsilon+\tau_\sigma\,\frac{d^\alpha\varepsilon}{dt^\alpha}\Big), \tag{14}$$

where $\mu_0=\mu(\omega)\Big|_{\omega=0}$, $\mu_\infty=\lim\limits_{\omega\to\infty}\mu(\omega)$, $\tau_\varepsilon/\tau_\sigma=\mu_0/\mu_\infty$, and ω is the frequency of action on the sample.

The equations with the fractional derivatives (14) are conveniently solved using a Fourier transform:

$$\bar f(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt; \tag{15}$$

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar f(\omega)\exp(i\omega t)\,d\omega. \tag{16}$$

Using the rule of finding a fractional derivative (see (11) and [19]), from (15) we obtain

$$\frac{d^\alpha f(t)}{dt^\alpha}=\frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega)^\alpha\bar f(\omega)\exp(i\omega t)\,d\omega. \tag{17}$$

Applying the Fourier transform (15) to (14) with allowance for (17), we obtain $\bar\sigma+(i\omega\tau)^\alpha\bar\sigma=2\mu_0(\bar\sigma+(i\omega\tau)^\alpha\bar\varepsilon)$, whence $\mu(i\omega)=\mu_\infty-(\mu_\infty-\mu_0)/[1+(i\omega\tau_\varepsilon)^\alpha]$. Taking into account that $\mu(i\omega)=\mu+i\mu'$ [$\mu=\operatorname{Re}\mu(i\omega)$, $\mu'=\operatorname{Im}\mu(i\omega)$], we have

$$\begin{aligned}\frac{\mu_\infty-\mu}{\mu_\infty-\mu_0}&=\frac{1+(\omega\tau)^\alpha\cos(\pi\alpha/2)}{1+2(\omega\tau)^\alpha[\cos(\pi\alpha/2)+(\omega\tau)^\alpha]},\\ \frac{\mu_\infty-\mu'}{\mu_\infty-\mu_0}&=\frac{(\omega\tau)^\alpha\sin(\pi\alpha/2)}{1+2(\omega\tau)^\alpha[\cos(\pi\alpha/2)+(\omega\tau)^\alpha]}.\end{aligned} \tag{18}$$

The parameter α is related to the fractional dimension of the fractal set and is a characteristic of localization (diffusion) of the relaxation spectrum.

Figure 2a and b shows dispersion dependences of the real μ (a) and imaginary μ' (b) parts of the relative shear modulus of a viscoelastic medium on log t^* ($t^*=\omega\tau$) [curve 1 refers to $\alpha=d_f=0.63$ (Cantor set) and curve 2 refers to $\alpha=0.9$].

To determine the dependence of the distribution density of the relaxation spectrum and the parameter α, we consider the following problem.

Let the strains be described by the step function $\varepsilon=\varepsilon_0\eta(t)$, $d\varepsilon/dt=\varepsilon_0\delta(t)$, where $\eta(t)$ is the Heaviside function and $\delta(t)$ is the Dirac delta function. Then, for the stress σ, we can write [22] $\sigma=2\mu(t)\varepsilon_0$, $\mu(t)=\mu_0+\Phi_\mu(t)$, where, for a Ziner medium, the function $\Phi_\mu(t)$ has the form

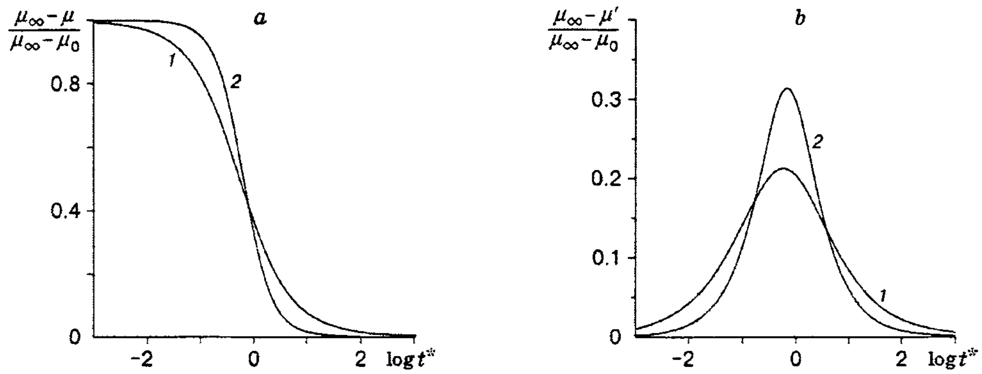

Fig. 2

$$\Phi_\mu(t) = (\mu_\infty - \mu_0)\exp(-t/\tau_\varepsilon). \tag{19}$$

If the medium is described by a discontinuous distribution of the relaxation time τ_i, by analogy with (19), we write

$$\Phi_\mu(t) = (\mu_\infty - \mu_0)\sum_i N_i \exp(-t/\tau_\varepsilon^i), \qquad \sum_i n_i = 1. \tag{20}$$

Conversion from the integral sum to the integral (12) according to (20) yields

$$\Phi_\mu(t) = \int_{-\infty}^{\infty} f_1(\tau)\exp(-t/\tau_\varepsilon)(d\tau)^\alpha, \tag{21}$$

where $f_1(\tau)$ is the density of the relaxation spectrum. Formula (21) can be brought to the form

$$\Phi_\mu(t) = \int_{-\infty}^{\infty} f(\tau)\exp(-t/\tau_\varepsilon)(d\ln\,\tau)^\alpha.$$

For the relaxation time distribution function $f(\tau)$, the normalization condition is given by

$$\int_{-\infty}^{\infty} f(\tau)(d\ln\,\tau)^\alpha = \mu_\infty - \mu_0.$$

Thus, the relation between the shear modulus $\mu(t)$ and the distribution function $f(\tau)$ becomes

$$\mu(t) = \mu_0 + \int_{-\infty}^{\infty} f(\tau)\exp(-t/\tau_\varepsilon)\,(d\ln\,\tau)^\alpha.$$

If the Fourier transform $\mu(t)$ is known, the Fourier transform of the function $f(\tau)$ has the form [21]

$$\bar{f}(1/\omega) = \pm(1/\pi)\mathrm{Im}\,\mu(\omega\exp(\pm i\pi)). \tag{22}$$

Substituting the values of $\mu'(t)$ from (18) into (22), it is possible to determine $f(\tau)$ for a Ziner medium:

$$f(\tau) = \frac{(\mu_\infty - \mu_0)\sin(\alpha\pi)}{2\pi\{\cosh[\ln(\alpha(\tau/\tau_\varepsilon))] + \cos(\alpha\pi)\}}.$$

The normalized density distribution of a Ziner medium $f_0(\tau) = f(\tau)/(\mu_\infty - \mu_0)$, or

$$f_0(\tau) = \frac{\sin(\alpha\pi)}{2\pi\{\cosh[\alpha\ln(\tau/\tau_\varepsilon)] + \cos(\alpha\pi)\}}$$

for two dynamic states is shown in Fig. 3. For $\alpha = 0.9$ (curve 2 in Fig. 3), i.e., for a medium whose dynamics is close to the dynamics with continuous states, the function f_0 in semilogarithmic coordinates becomes the

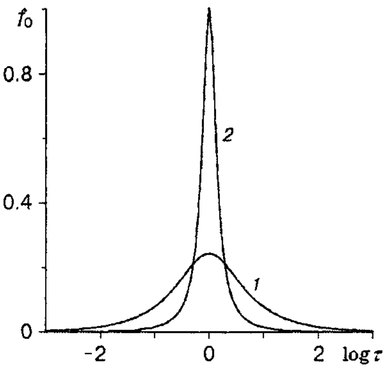

Fig. 3

Dirac delta function with $\alpha = 1$. For a medium with random dynamics that generates states with fractal dimension $\alpha = d_f = 0.63$ (curve 1 in Fig. 3), which is equal to the dimension of Cantor "dust," the function f_0 has a diffused spectrum, i.e., the order of the fractional derivative α can be treated as a characteristic of the diffusion of the relaxation spectrum.

The relation between the dispersion of the relaxation time of random dynamics γ^2 and the parameter α has the form

$$\gamma^2 = \int_{-\infty}^{\infty} \ln^2(\tau/\tau_\varepsilon) f_0(\tau)\, d\ln\tau = \frac{\pi^2}{3}\frac{1-\alpha^2}{\alpha^2},$$

i.e., $\gamma^2 = (\pi^2/3)(1 - d_f^2)/d_f^2$.

Thus, the relationship between the fractal dimension, the fractional derivative, and the diffusion of the relaxation spectrum is established from the viscoelastic properties of a disordered fractal medium.

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